CSS

GS-2013
(Computer & Systems Sciences)
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **COMPUTER & SYSTEMS SCIENCES - December 9, 2012**
Duration : Three hours (3 hours)

Name : _____ Ref. Code : _____

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. <u>Do not mark more than one circle for any question</u> : this will be treated as a wrong answer.

3. This question paper consists of three (3) parts. **Part-A** contains twenty (20) questions and **must be attempted** by all candidates. **Part-B & Part-C** contain twenty (20) questions each, directed towards candidates for (B) Computer Science and (C) Systems Science (including Communications & Math Finance), respectively. **STUDENTS MAY ATTEMPT EITHER PART-B OR PART-C.   In case, a student attempts both Parts B & C (no extra time will be given) and qualifies for interview in both B & C, he/she will have opportunity to be interviewed in both areas.** All questions carry equal marks. A correct answer for a question will give you +4 marks, a wrong answer will give you -1 mark, and a question not answered will not get you any marks.

4. We advise you to first mark the correct answers in the QUESTION SHEET and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

Part A
Common Questions

1. An infinite two-dimensional pattern is indicated below.
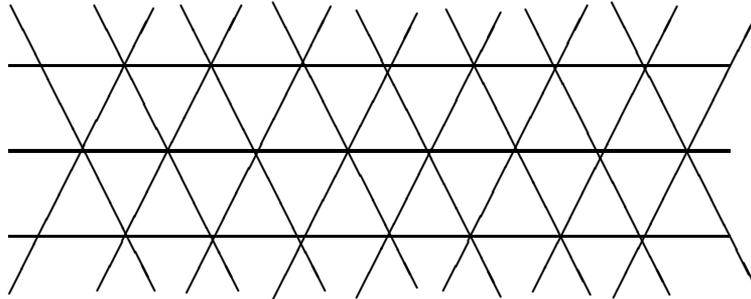
The smallest closed figure made by the lines is called a unit triangle. Within every unit triangle, there is a mouse. At every vertex there is a *laddoo*. What is the average number of *laddoos* per mouse?

(a) 3

(b) 2

(c) 1

(d) $\frac{1}{2}$ ✔

(e) $\frac{1}{3}$

2. Consider the following two types of elections to determine which of two parties A and B forms the next government in the 2014 Indian elections. Assume for simplicity an Indian population of size 545545 (= 545*1001). There are only two parties A and B and every citizen votes.

TYPE C: The country is divided into 545 constituencies and each constituency has 1001 voters. Elections are held for each constituency and a party is said to win a constituency if it receive a majority of the vote in that constituency. The party that wins the most constituencies forms the next government.

TYPE P: There are no constituencies in this model. Elections are held throughout the country and the party that wins the most votes (among 545545 voters) forms the government.

Which of the following is true?

(a) If a party forms the govt. by election TYPE C winning at least two-thirds of the constituencies, then it will also form the govt. by election TYPE P.

(b) If a party forms the govt. by election TYPE C, then it will also form the govt. by election TYPE P.

(c) If a party forms the govt. by election TYPE P, then it will also form the govt. by election TYPE C.

(d) All of the above

(e) None of the above ✔

3. Three candidates, Amar, Birendra and Chanchal stand for the local election. Opinion polls are conducted and show that fraction a of the voters prefer Amar to Birendra, fraction b prefer Birendra to Chanchal and fraction c prefer Chanchal to Amar. Which of the following is impossible?

(a) $(a, b, c) = (0.51, 0.51, 0.51)$;

(b) $(a, b, c) = (0.61, 0.71, 0.67)$;

(c) $(a, b, c) = (0.68, 0.68, 0.68)$; ✔

(d) $(a, b, c) = (0.49, 0.49, 0.49)$;

(e) None of the above.

4. A biased coin is tossed repeatedly. Assume that the outcomes of different tosses are independent and probability of heads is $2/3$ in each toss. What is the probability of obtaining an even number of heads in 5 tosses, zero being treated as an even number?

(a) $121/243$

(b) $122/243$ ✔

(c) $124/243$

(d) $125/243$

(e) $128/243$

5. The late painter Maqbool Fida Husain once coloured the surface of a huge hollow steel sphere, of radius 1 metre, using just two colours, red and blue. As was his style however, both the red and blue areas were a bunch of highly irregular disconnected regions. The late sculptor Ramkinkar Baij then tried to fit in a cube inside the sphere, the eight vertices of the cube touching only red coloured parts of the surface of the sphere. Assume $\pi = 3.14$ for solving this problem. Which of the following is true?

(a) Baij is bound to succeed if the area of the red part is 10 sq. metres;

(b) Baij is bound to fail if the area of the red part is 10 sq. metres;

(c) Baij is bound to fail if the area of the red part is 11 sq. metres;

(d) Baij is bound to succeed if the area of the red part is 11 sq. metres; ✔

(e) None of the above.

6. You are lost in the national park of Kabrastan. The park population consists of tourists and kabrastanis. Tourists comprise two-thirds of the population the park, and give a correct answer to requests for directions with probability $3/4$. The air of Kabrastan has an amnesaic quality however, and so the answers to repeated questions to tourists are independent, even if the question and the person are the same. If you ask a Kabrastani for directions, the answer is always wrong.

Suppose you ask a randomly chosen passer-by whether the exit from the park is East or West. The answer is East. You then ask the same person again, and the reply is again East. What is the probability of East being correct?

(a) $1/4$

(b) $1/3$

(c) $1/2$ ✔

(d) $2/3$

(e) $3/4$

7. For any complex number z, arg z defines its phase, chosen to be in the interval $0 \leq arg\, z < 360^o$. If z_1, z_2 and z_3 are three complex numbers with the same modulus but different phases (arg $z_3 <$ arg $z_2 <$ arg $z_1 < 180^o$), then the quantity

$$\frac{arg\,(z_1/z_2)}{arg\,[(z_1 - z_3)/(z_2 - z_3)]}$$

is a constant, and has the value

(a) 2 ✔

(b) $\frac{1}{3}$

(c) 1

(d) 3

(e) $\frac{1}{2}$

8. Find the sum of the infinite series

$$\frac{1}{1 \times 3 \times 5} + \frac{1}{3 \times 5 \times 7} + \frac{1}{5 \times 7 \times 9} + \frac{1}{7 \times 9 \times 11} + \text{.......}$$

(a) ∞

(b) $\frac{1}{2}$

(c) $\frac{1}{6}$

(d) $\frac{1}{12}$ ✔

(e) $\frac{1}{14}$

3

9. There are n kingdoms and $2n$ champions. Each kingdom gets 2 champions. The number of ways in which this can be done is:

(a) $\dfrac{(2n)!}{2^n}$ ✓

(b) $\dfrac{(2n)!}{n!}$

(c) $\dfrac{(2n)!}{2^n \cdot n!}$

(d) $n!/2$

(e) None of the above

10. Three men and three *rakhsasas* arrive together at a ferry crossing to find a boat with an oar, but no boatman. The boat can carry one or at the most two persons, for example, one man and one *rakhsasa*, and each man or *rakhsasa* can row. But if at any time, on any bank, (including those who maybe are in the boat as it touches the bank) *rakhsasas* outnumber men, the former will eat up the latter. If all have to go to the other side without any mishap, what is the minimum number of times that the boat must cross the river?

(a) 7

(b) 9

(c) 11 ✓

(d) 13

(e) 15

11. Let there be a pack of 100 cards numbered 1 to 100. The ith card states: "There are at most $i-1$ true cards in this pack". Then, how many cards of the pack contain TRUE statements?

(a) 0

(b) 1

(c) 100

(d) 50 ✓

(e) None of the above

12. Among numbers 1 to 1000 how many are divisible by 3 or 7?

(a) 333

(b) 142

(c) 475

(d) 428 ✓

(e) None of the above

13. Doctors A and B perform surgery on patients in stages III and IV of a disease. Doctor A has performed a 100 surgeries (on 80 stage III and 20 stage IV patients) and 80 out of her 100 patients have survived (78 stage III and 2 stage IV survivors). Doctor B has also performed 100 surgeries (on 50 stage III and 50 stage IV patients). Her success rate is 60/100 (49 stage III survivors and 11 stage IV survivors). A patient has been advised that she is equally likely to be suffering from stage III or stage IV of this disease. Which doctor would you recommend to this patient and why?

(a) Doctor A since she has a higher success rate

(b) Doctor A since she specializes in stage III patients and the success of surgery in stage IV patients is anyway too low

(c) Doctor B since she has performed more stage IV surgeries

(d) Doctor B since she appears to be more successful ✓

(e) There is not enough data since the choice depends on the stage of the disease the patient is suffering from

14. An unbiased die is thrown n times. The probability that the product of numbers would be even is

(a) $1/(2n)$

(b) $1/[(6n)!]$

(c) $1 - 6^{-n}$

(d) 6^{-n}

(e) None of the above ✓

15. Let $\text{sgn}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$.

What is the value of the summation

$$\sum_{i=0}^{50} \text{sgn}(\, (2i-1)(2i-3)\cdots(2i-99)\,)$$

(a) 0

(b) -1

(c) $+1$ ✔

(d) 25

(e) 50

16. The minimum of the function $f(x) = x\log_e(x)$ over the interval $[1/2, \infty)$ is

(a) 0

(b) $-e$

(c) $-\log_e(2)/2$ ✔

(d) $-1/e$

(e) None of the above

17. A stick of unit length is broken into two at a point chosen at random. Then, the larger part of the stick is further divided into two parts in the ratio 4:3. What is the probability that the three sticks that are left CANNOT form a triangle?

(a) 1/4 ✔

(b) 1/3

(c) 5/6

(d) 1/2

(e) $\log_e(2)/2$.

18. Consider three independent uniformly distributed (taking values between 0 and 1) random variables. What is the probability that the middle of the three values (between the lowest and the highest value) lies between a and b where $0 \le a < b \le 1$.

   (a) $3(1-b)a(b-a)$

   (b) $3((b-a) - (b^2 - a^2)/2)$

   (c) $6(1-b)a(b-a)$

   (d) $(1-b)a(b-a)$

   (e) $6((b^2 - a^2)/2 - (b^3 - a^3)/3)$. ✔

19. Consider a sequence of numbers $(\epsilon_n : n = 1, 2, \ldots)$, such that $\epsilon_1 = 10$ and

$$\epsilon_{n+1} = \frac{20\epsilon_n}{20 + \epsilon_n}$$

for $n \ge 1$. Which of the following statements is true?
   Hint: Consider the sequence of reciprocals.

   (a) The sequence $(\epsilon_n : n = 1, 2, \ldots)$ converges to zero. ✔

   (b) $\epsilon_n \ge 1$ for all n.

   (c) The sequence $(\epsilon_n : n = 1, 2, \ldots)$ is decreasing and converges to 1.

   (d) The sequence $(\epsilon_n : n = 1, 2, \ldots)$ is decreasing and then increasing. Finally it converges to 1.

   (e) None of the above

20. Consider a well functioning clock where the hour, minute and the seconds needles are exactly at zero. How much time later will the minutes needle be exactly one minute ahead (1/60 th of the circumference) of the hours needle and the seconds needle again exactly at zero?

   Hint: When the desired event happens both the hour needle and the minute needle have moved an integer multiple of 1/60 th of the circumference.

   (a) 144 minutes

   (b) 66 minutes

   (c) 96 minutes

   (d) 72 minutes

   (e) 132 minutes ✔

Part B
Computer Science Questions

1. Let $G = (V, E)$ be a simple undirected graph on n vertices. A *colouring* of G is an assignment of colours to each vertex such that endpoints of every edge are given different colours. Let $\chi(G)$ denote the *chromatic number* of G, i.e., the minimum number of colours needed for a valid colouring of G. A set $B \subseteq V$ is an *independent set* if no pair of vertices in B is connected by an edge. Let $a(G)$ be the number of vertices in a largest possible independent set in G. In the absence of any further information about G we can conclude

(a) $\chi(G) \geq a(G)$

(b) $\chi(G) \leq a(G)$

(c) $a(G) \geq n/\chi(G)$ ✔

(d) $a(G) \leq n/\chi(G)$

(e) None of the above

2. Consider polynomials in a single variable x of degree d. Suppose $d < n/2$. For such a polynomial $p(x)$, let C_p denote the n-tuple $(p(i))_{1 \leq i \leq n}$. For any two such distinct polynomials p, q, the number of coordinates where the tuples C_p, C_q differ is

(a) at most d

(b) at most $n - d$

(c) between d and $n - d$

(d) at least $n - d$ ✔

(e) none of the above

3. How many 4×4 matrices with entries from $\{0, 1\}$ have odd determinant?
Hint: Use modulo 2 arithmetic.

(a) 20160 ✔

(b) 32767

(c) 49152

(d) 57343

(e) 65520

4. A set S together with partial order $\ll$ is called a well order if it has no infinite descending chains, i.e. there is no infinite sequence $x_1, x_2, \ldots$ of elements from S such that $x_{i+1} \ll x_i$ and $x_{i+1} \neq x_i$ for all i. Consider the set of all words (finite sequences of letters a-z), denoted by W, in dictionary order.

(a) Between "aa" and "az" there are only 24 words

(b) Between "aa" and "az" there are only 2^{24} words

(c) W is not a partial order

(d) W is a partial order but not a well order

(e) W is a well order ✓

5. Given a weighted directed graph with n vertices where edge weights are integers (positive, zero, or negative), determining whether there are paths of arbitrarily large weight can be performed in time

(a) $O(n)$

(b) $O(n \cdot log(n))$ but not $O(n)$

(c) $O(n^{1.5})$ but not $O(n \log n)$

(d) $O(n^3)$ but not $O(n^{1.5})$ ✓

(e) $O(2^n)$ but not $O(n^3)$

6. Let L and L' be languages over the alphabet Σ. The left quotient of L by L' is

$$L/L' \stackrel{\text{def}}{=} \{w \in \Sigma^* : wx \in L \text{ for some } x \in L'\}.$$

Which of the following is true?

(a) If L/L' is regular then L' is regular

(b) If L is regular then L/L' is regular ✓

(c) If L/L' is regular then L is regular

(d) L/L' is a subset of L

(e) If L/L' and L' are regular, then L is regular

7. Which of the following is not implied by $\mathbf{P} = \mathbf{NP}$?

(a) 3SAT can be solved in polynomial time

(b) Halting problem can be solved in polynomial time ✓

(c) Factoring can be solved in polynomial time.

(d) Graph isomorphism can be solved in polynomial time.

(e) Travelling salesman problem can be solved in polynomial time.

8. Which one of the following languages over the alphabet $\{0, 1\}$ is regular?

(a) The language of balanced parentheses where $0, 1$ are thought of as $(,)$ respectively

(b) The language of palindromes, i.e., bit strings x that read the same from left to right as well as right to left

(c) $L = \{0^{m^2} : 3 \leq m\}$

(d) The Kleene closure L^*, where L is the language in (c) above ✔

(e) $\{0^m 1^n \mid 1 \leq m \leq n\}$

9. Suppose n straight lines are drawn on a plane. When these lines are removed, the plane falls apart into several connected components called *regions*. A region R is said to be *convex* if it has the following property: whenever two points are in R, then the entire line segment joining them is in R. Suppose no two of the n lines are parallel. Which of the following is true?

(a) $O(n)$ regions are produced, and each region is convex

(b) $O(n^2)$ regions are produced but they need not all be convex

(c) $O(n^2)$ regions are produced, and each region is convex ✔

(d) $O(n \log n)$ regions are produced, but they need not all be convex

(e) All regions are convex but there may be exponentially many of them

10. Let m, n be positive integers with m a power of 2. Let $s = 100n^2 \log m$. Suppose $S_1, S_2, \ldots, S_m$ are subsets of $\{1, 2, \ldots, s\}$ such that $|S_i| = 10n \log m$ and $|S_i \cap S_j| \leq \log m$ for all $1 \leq i < j \leq m$. Such a collection of sets $S_1, \ldots, S_m$ is an example of a so-called *Nisan-Wigderson design*. We now consider the *set membership problem*, where we have to store an arbitrary subset $T \subseteq \{1, 2, \ldots, m\}$, $|T| = n$ as an array A of s bits so that given any integer x, $1 \leq x \leq m$, we can discover whether $x \in T$ by reading only *one* bit of A. Consider the following strategy to solve this problem. Array A is initialised to all zeroes. Given the set T to be stored, we put a one in all the locations of A indexed by the union $\cup_{t \in T} S_t$. Now, given the integer x, we read a random location in A from S_x and declare that $x \in T$ iff the bit in that location is one. This strategy gives the correct answer with probability

(a) 1 if $x \in T$ and at most 0.1 if $x \notin T$.

(b) at least 0.9 if $x \in T$ and at most 0.1 if $x \notin T$.

(c) at least 0.9 if $x \in T$ and at least 0.9 if $x \notin T$.

(d) 1 if $x \in T$ and at least 0.9 if $x \notin T$. ✔

(e) at least 0.9 if $x \in T$ and 1 if $x \notin T$.

11. Which of the following statements is FALSE?

(a) The intersection of a context free language with a regular language is context free.

(b) The intersection of two regular languages is regular.

(c) The intersection of two context free languages is context free ✔

(d) The intersection of a context free language and the complement of a regular language is context free.

(e) The intersection of a regular language and the complement of a regular language is regular.

12. It takes $O(n)$ time to find the median in a list of n elements, which are not necessarily in sorted order while it takes only $O(1)$ time to find the median in a list of n sorted elements. How much time does it take to find the median of $2n$ elements which are given as two lists of n sorted elements each?

(a) $O(1)$

(b) $O(\log n)$ but not $O(1)$ ✔

(c) $O(\sqrt{n})$ but not $O(\log n)$

(d) $O(n)$ but not $O(\sqrt{n})$

(e) $O(n \log n)$ but not $O(n)$

13. Given a binary tree of the following form and having n nodes, the height of the tree is

(a) $\Theta(\log\ n)$

(b) $\Theta(n)$

(c) $\Theta(\sqrt{n})$ ✔

(d) $\Theta(n/\log n)$

(e) None of the above

14. Assume a demand paged memory system where ONLY THREE pages can reside in the memory at a time. The following sequence gives the order in which the program references the pages.

$$1, 3, 1, 3, 4, 2, 2, 4$$

Assume that least frequently used page is replaced when necessary. If there is more than one least frequently used pages then the least recently used page among them is replaced. During the program's execution, how many times will the pages 1,2,3 and 4 be brought to the memory?

(a) 2,2,2,2 times, respectively

(b) 1,1,1,2 times, respectively ✔

(c) 1,1,1,1 times, respectively

(d) 2,1,2,2 times, respectively

(e) None of the above

15. Let G be an undirected graph with n vertices. For any subset S of vertices, the set of neighbours of S consists of the union of S and the set of vertices S' that are connected to some vertex in S by an edge of G. The graph G has the nice property that every subset of vertices S of size at most $n/2$ has at least $1.5|S|$-many neighbours. What is the length of a longest path in G?

(a) $O(1)$

(b) $O(\log \log n)$ but not $O(1)$

(c) $O(\log n)$ but not $O(\log \log n)$ ✔

(d) $O(\sqrt{n})$ but not $O(\log n)$

(e) $O(n)$ but not $O(\sqrt{n})$

16. Consider a function $T_{k,n} : \{0,1\}^n \to \{0,1\}$ which returns 1 iff at least k of its n inputs are 1. Formally, $T_{k,n}(x) = 1$ iff $\sum_1^n x_i \geq k$. Let $y \in \{0,1\}^n$ be such that y has exactly k ones. Then, the function $T_{k,n-1}(y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ (where y_i is omitted) is equivalent to

(a) $T_{k-1,n}(y)$

(b) $T_{k,n}(y)$

(c) y_i

(d) $\neg y_i$ ✔

(e) None of the above

17. In a connected weighted graph with n vertices, all the edges have distinct positive integer weights. Then, the maximum number of minimum weight spanning trees in the graph is

(a) 1 ✓

(b) n

(c) equal to number of edges in the graph

(d) equal to maximum weight of an edge of the graph

(e) n^{n-2}

18. Let S be a set of numbers. For $x \in S$, the *rank* of x is the number of elements in S that are less than or equal to x. The procedure **Select**(S, r) takes a set S of numbers and a rank r ($1 \le r \le |S|$) and returns the element in S of rank r. The procedure **MultiSelect**(S, R) takes a set of numbers S and a list of ranks $R = \{r_1 < r_2 < \cdots < r_k\}$, and returns the list $\{x_1 < x_2 < \cdots < x_k\}$ of elements of S, such that the rank of x_i is r_i. Suppose there is an implementation for **Select**(S, r) that uses at most (constant $\cdot |S|$) binary comparisons between elements of S. The minimum number of comparisons needed to implement **MultiSelect**(S, R) is

(a) constant $\cdot |S| \log |S|$

(b) constant $\cdot |S|$

(c) constant $\cdot |S||R|$

(d) constant $\cdot |R| \log |S|$

(e) constant $\cdot |S|(1 + \log |R|)$ ✓

19. In a relational database there are three relations:

- *Customers* = $C(\mathsf{CName})$,

- *Shops* = $S(\mathsf{SName})$,

- *Buys* = $B(\mathsf{CName}, \mathsf{SName})$.

Which of the following relational algebra expressions returns the names of shops that have no customers at all? [Here Π is the projection operator.]

(a) $\Pi_{\mathsf{SName}} B$

(b) $S - B$

(c) $S - \Pi_{\mathsf{SName}} B$ ✓

(d) $S - \Pi_{\mathsf{SName}}((C \times S) - B)$

(e) None of the above

20. Suppose n processors are connected in a linear array as shown below. Each processor has a number. The processors need to exchange numbers so that the numbers eventually appear in ascending order (the processor P1 should have the minimum value and the the processor Pn should have the maximum value).
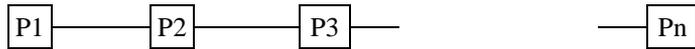
The algorithm to be employed is the following. Odd numbered processors and even numbered processors are activated alternate steps; assume that in the first step all the even numbered processors are activated. When a processor is activated, the number it holds is compared with the number held by its right-hand neighbour (if one exists) and the smaller of the two numbers is retained by the activated processor and the bigger stored in its right-hand neighbour.

How long does it take for the processors to sort the values?

(a) $n \log n$ steps

(b) n^2 steps

(c) n steps ✓

(d) $n^{1.5}$ steps

(e) the algorithm is not guaranteed to sort

Part C
Systems Science Questions

1. The unit step response of a discrete-time, linear, time-invariant system is

$$y[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \text{ and } n \text{ even} \\ -1, & n \geq 0 \text{ and } n \text{ odd} \end{cases}$$

Which of the following statements about the system is true?

(a) the system is anti-causal

(b) the system is memoryless

(c) the system is bounded-input, bounded-output (BIBO) stable

(d) there is not enough information to determine BIBO stability

(e) none of the above ✔

2. The output $\{y(n)\}$ of a discrete time system with input $\{x(n)\}$ is given by

$$y(n) = \sum_{k=0}^{N-1} a^k x(n-k).$$

The difference equation for the *inverse system* is given by

(a) $y(n) = x(n) - ax(n-1)$

(b) $y(n) - a^N y(n-N) = x(n) - ax(n-1)$ ✔

(c) If $|a| < 1$, then the answer is (a) above, otherwise the inverse does not exist

(d) If $|a| < 1$, then the answer is (b) above, otherwise the inverse does not exist

(e) None of the above

3. X and Y are jointly Gaussian random variables with zero mean.
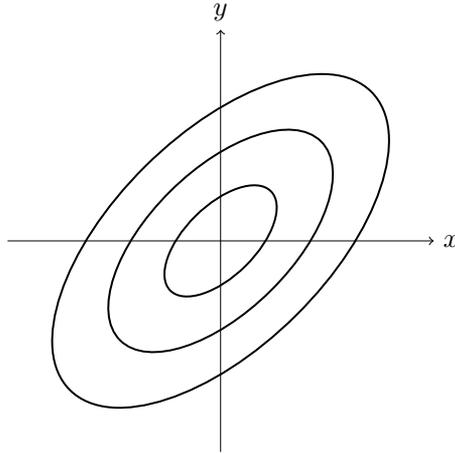
A constant-pdf contour is where the joint density function takes on the same value. If the constant-pdf contours of X, Y are as shown above, which of the following could their covariance matrix $\mathbf{K}$ be:

(a) $\mathbf{K} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ ✔️

(b) $\mathbf{K} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

(c) $\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(e) $\mathbf{K} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 2 \end{bmatrix}$

4. Consider a fair coin that has probability 1/2 of showing heads (H) in a toss and 1/2 of showing tails (T). Suppose we independently flip a fair coin over and over again. What is the probability that HT sequence occurs before TT?

(a) 3/4 ✔️

(b) 1/2

(c) 2/3

(d) 1/3

(e) 1/4

5. Let $x(n) = \sin(2\pi kn/N)$, $n = 0, 1, ..., N - 1$, where $2k \neq N$ and $0 < k \leq N - 1$. Then the circular convolution of $\{x(n)\}$ with itself is

(a) $N \cos(4\pi kn/N)$

(b) $N \sin(4\pi kn/N)$

(c) $-N \cos(2\pi kn/N)/2$ ✔

(d) $-N \sin(2\pi kn/N)/2$

(e) None of the above

6. The two-dimensional Fourier transform of a function $f(t, s)$ is given by

$$F(\omega, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, s) \exp(-j\omega t) \exp(-j\theta s) dt ds.$$

Let $\delta(t)$ be the delta function and let $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t \geq 0$. If the Fourier transform is $F(\omega, \theta) = \delta(\omega - \theta)/(j\omega + 1)$, then $f(t, s)$ equals a constant multiple of

(a) $\exp(-(t - s))u(t - s)$

(b) $\exp(-(t + s))u(t + s)$ ✔

(c) $\exp(-t)u(t)\delta(s)$

(d) $\exp(-t)\delta(t + s)$

(e) None of the above

7. The Z-transform of $\{x(n)\}$ is defined as $X(z) = \sum_n x(n)z^{-n}$ (for those z for which the series converges). Let $u(n) = 1$ for $n \geq 0$ and $u(n) = 0$ for $n < 0$. The inverse Z-transform of $X(z) = \log(1 - az)$, $|z| < 1/a$ is

(a) $a^n u(n - 1)/n$

(b) $a^{-n} u(-n - 1)/n$ ✔

(c) If $|a| < 1$, then the answer is (a), otherwise the inverse is not well-defined

(d) If $|a| < 1$, then the answer is (b), otherwise the inverse is not well-defined

(e) None of the above

8. The following circuit with an ideal operational amplifier is
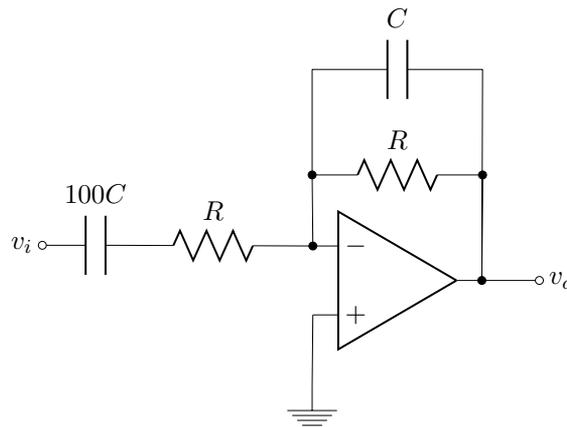
(a) A low pass filter

(b) A high pass filter

(c) A bandpass filter ✔

(d) A bandstop filter

(e) An all pass amplifier

9. Let X and Y be two zero mean independent continuous random variables. Let $Z_1 = \max(X, Y)$, and $Z_2 = \min(X, Y)$. Then which of the following is TRUE.

(a) Z_1 and Z_2 are uncorrelated. ✔

(b) Z_1 and Z_2 are independent.

(c) $P(Z_1 = Z_2) = \frac{1}{2}$.

(d) Both (a) and (c)

(e) Both (a) and (b)

10. Consider the following series of square matrices:

$$H_1 = [1],$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and for $k = 2, 3, ...,$ the $2^k \times 2^k$ matrix H_{2^k} is recursively defined as

$$H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}.$$

What is $|\det(H_{2^k})|$?   (*Hint:* What is $H_{2^k} H_{2^k}^T$?)

(a) 0

(b) 2^k

(c) $2^{k/2}$

(d) $2^{k2^{k-1}}$ ✔

(e) 2^{k2^k}

11.   Two matrices A and B are called similar if there exists another matrix S such that $S^{-1}AS = B$. Consider the statements:

(I) If A and B are similar then they have identical rank.

(I) If A and B are similar then they have identical trace.

(III) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ are similar.

Which of the following is TRUE.

(a) Only I.

(b) Only II.

(c) Only III

(d) Both I and II but not III.

(e) All of I, II and III. ✔

12.   Let A be a Hermitian matrix and let I be the Identity matrix with same dimensions as A. Then for a scalar $\alpha > 0$, $A + \alpha I$ has

(a) the same eigenvalues as of A but different eigenvectors

(b) the same eigenvalues and eigenvectors as of A

(c) the eigenvalues smaller than those of A and same eigenvectors as of A

(d) eigenvalues and eigenvectors with no relation to those of A

(e) None of the above ✔

13. Let A be a square matrix and x be a vector whose dimensions match A. Let $B^\dagger$ be the conjugate transpose of B. Then which of the following is not true:

(a) $x^\dagger A^2 x$ is always non-negative ✔

(b) $x^\dagger A x$ could be zero

(c) $x^\dagger A x$ could be complex

(d) If $A = A^\dagger$ then $x^\dagger A x$ is real

(e) If $A = A^\dagger$ then $x^\dagger A y$ is complex for some vector y with same dimensions as x

14. X, Y, Z are integer valued random variables with the following two properties:

(i) X and Y are independent.

(ii) For all integer x, conditioned on the event $\{X = x\}$, we have that Y and Z are independent (in other words, conditioned on X, the random variables Y and Z are independent).

Which of the following can we infer?

(a) The random variables X and Z are independent

(b) Conditioned on Y, the random variables X and Z are independent

(c) Conditioned on Z, the random variables X and Y are independent ✔

(d) All of the above

(e) None of the above

15. Consider a sequence of non-negative numbers $\{x_n : n = 1, 2, \ldots\}$. Which of the following statements cannot be true?

(a) $\sum_{n=1}^\infty x_n = \infty$ but x_n decreases to zero as n increases.

(b) $\sum_{n=1}^\infty x_n < \infty$ and each $x_n > 0$ for each n.

(c) $\sum_{n=1}^\infty x_n = \infty$ and $x_n \geq 0.01$ infinitely often.

(d) $\sum_{n=1}^\infty x_n = \infty$ and each $x_n \leq 1/n^2$. ✔

(e) $\sum_{n=1}^\infty x_n < \infty$ and each $x_n > x_{n+1}$.

16. A surprise quiz contains three multiple choice questions; question 1 has 3 suggested answers, question 2 has four, and question 3 has two. A completely unprepared student decides to choose the answers at random. If X is the number of questions the student answers correctly, the expected number of correct answers is

(a) 15/12

(b) 7/12

(c) 13/12 ✔

(d) 18/12

(e) None of the above

17. Consider four coins, three of which are fair, that is they have heads on one side and tails on the other and both are equally likely to occur in a toss. The fourth coin has heads on both sides. Given that one coin amongst the four is picked at random and is tossed, and the outcome is seen to be head, what is the probability that its other side is tails?

(a) 1/2

(b) 3/8

(c) 3/5 ✔

(d) 3/4

(e) 5/7

18. Consider a coin tossing game between Santa and Banta. Both of them toss two coins sequentially, first Santa tosses a coin then Banta and so on. Santa tosses a fair coin: Probability of heads is 1/2 and probability of tails is 1/2. Banta's coin probabilities depend on the outcome of Santa's toss. If Santa sees an outcome heads then Banta gets a coin whose probability of heads is 3/4, and of tails is 1/4. If Santa tosses a tail then Banta gets a coin whose probability of tossing a tails is 3/4 and of heads is 1/4.

What is the probability of the event that they both have one head and one tail in the two trials conducted by each of them?

(a) 1/2

(b) 5/16 ✔

(c) 3/16

(d) 1/4

(e) 1/3

19. Which of the following is true for polynomials defined over real numbers $\mathbb{R}$.

(a) Every odd degree polynomial has a real root. ✓

(b) Every odd degree polynomial has at least one complex root.

(c) Every even degree polynomial has at least one complex root.

(d) Every even degree polynomial has a real root.

(e) None of the above

20. A function $f : \mathbb{R} \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}, \alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$. Which of the following is not convex:

(a) x^2

(b) x^3 ✓

(c) x

(d) x^4

(e) e^x